Influence of the noise spectrum on the anomalous diffusion in a stochastic system

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We use an effective Markovian description to study the long-time behavior of a nonlinear second-order Langevin equation with a Gaussian noise. When dissipation is neglected, the energy of the system grows as with time a power law with an anomalous scaling exponent that depends both on the confining potential and on the high-frequency distribution of the noise. The asymptotic expression of the probability distribution function in phase space is calculated analytically. The results are extended to the case where small dissipative effects are taken into account.

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I. INTRODUCTION

The influence of a random perturbation on a dynamical system is a problem of interest in various fields of science and engineering [1–6]. The first example of a differential equation with stochastic terms appeared in Langevin's study of Brownian motion [7,8]. Langevin modeled the action of the solvent molecules on the Brownian particle as the sum of a deterministic viscous friction, proportional to the velocity of the Brownian particle, and of a random force of autocorrelation proportional to the temperature of the bath. Since then, it has been customary to add in the dynamical equations some phenomenological stochastic terms that describe random environmental loadings (e.g., the influence of a turbulent wind on a suspension bridge or the study of random parametric vibration of helicopter rotor blades in atmospheric turbulent flow [9,10]). Of particular interest is the determination of the energy flow into the system from external sources when the characteristic time of the parameters variations matches one of the natural frequencies of the system. Parametric resonance then occurs and the rate of increase in the amplitude is generally exponential leading to instability. The growth of the response is limited by various nonlinear effects.

Several methods have been developed to study random parametric vibrations [10]. One of the most efficient techniques is the averaging principle developed by Bogoliubov and Mitropol'skii for deterministic nonlinear vibrations [11], where rapidly fluctuating circular coordinates are averaged out leading to a set of effective dynamical equations for slow variables. This method was extended to stochastic systems by Stratonovich [6] and put on a rigorous mathematical basis by Khas'minskii [12] and by Papanicolaou and Kohler [13]. Since then, stochastic averaging has become a powerful method [14,15] (for a recent review, see [16] and references therein).

In a series of recent works [17–20], we studied the long-time behavior of the nonlinear oscillator subject to parametric noise. We showed that in the absence of dissipation, the nonlinear terms in the potential stiffness inhibit the exponential growth of the amplitude. The observables of the system (the amplitude, the momentum, and the energy) rather display power-law scalings with anomalous diffusion exponents. When the parametric noise is a Gaussian white noise (i.e., it has a vanishingly small correlation time), the averaging method applied to the energy envelope [6,10,21–23] allows us to calculate analytically the time-asymptotic probability distribution function (PDF) of the system in phase space [17]; knowing this PDF, the scaling exponents and the corresponding prefactors are readily deduced. However, for colored noise a competition between conflicting time scales occurs. In fact, the nonlinear oscillator has an amplitude-dependent intrinsic frequency that increases with the amplitude. As the oscillator absorbs energy from its environment, its amplitude grows and, at a certain stage, the intrinsic period becomes smaller than the correlation time of the noise. This corresponds to a cross-over regime at which the correlation time of the noise ceases to be the smallest time scale in the system. The scaling laws that govern the growth of amplitude, momentum, and energy differ from those calculated for a white noise. Thus, when the amplitude of the oscillator is small its intrinsic period is large and the noise appears as if it was white and white-noise exponents prevail; but at large amplitudes, the scaling regime changes and new exponents appear. Because of these conflicting time scales, the averaging technique is difficult to implement for a colored noise. At the lowest order, the noise itself is averaged out and the energy transfer stops at the cross-over time. Therefore, one has to perform averaging at higher orders. When the colored noise is an Ornstein-Uhlenbeck (OU) process the calculations can be carried out by a second-order averaging, which requires rather tedious mathematical manipulations [18,19]. It is also possible to calculate the crossover between the white noise and the Ornstein-Uhlenbeck scaling regimes. The averaging method works at second order for the Ornstein-Uhlenbeck noise because its time derivative is a white noise. However, if the random excitation is generated from a white noise through a differential equation on the order of n, one has to perform averaging at $(n+1)$th order and, in practice, the calculations are intractable.

In the present work, we follow an entirely different approach to study the nonlinear oscillator subject to a parametric Gaussian noise with an arbitrary spectrum (with the assumption that the spectrum decays as a power law at high frequencies). We shall use an effective coarse-grained Markovian description of the dynamics, following a technique developed by Carmeli and Nitzan [24] (see also [23,25,26] for a similar approach). This technique will allow us to calculate analytically the asymptotic PDF which leads to the

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formulae for the growth of the amplitude, of the momentum, and of the energy transfer. In particular, we shall prove that the scaling exponents depend both on the stiffness of the potential at infinity and on the smoothness of the random excitation; the smoother the noise (which corresponds to a faster decay of the power spectrum at high frequencies), the less efficient is the energy transfer from the bath to the oscillator. The method used here can be adapted both to additive and multiplicative noises and can also be used when a small friction is present. The system reaches at large times a nonequilibrium steady state in which physical observables do not grow anymore; the crossover from power-law growth to this steady state occurs when the rate of energy dissipation by friction matches that of energy absorption from the random environmental loading.

The outline of this work is as follows. In Sec. I, we define precisely the model we shall study. In Sec. II, we use the underlying integrability of the system to write exact dynamical equations in energy-angle variables and we use the coarse-grained Markovian description to derive an effective Fokker-Planck equation for the energy variable. In Sec. III, we derive explicit formulae for various cases: multiplicative or additive noise, with or without dissipation. This leads to a rather exhaustive description of all the different cases. In particular, we verify that this method allows us to recover the analytical results obtained previously for white and Ornstein-Uhlenbeck noises. The last section is devoted to concluding remarks.

II. NONLINEAR OSCILLATOR WITH PARAMETRIC NOISE

A paradigm for the study of interplay of noise and nonlinearity is the nonlinear oscillator subject to parametric random excitations,

$$\frac{d^2}{dt^2}x(t) + \gamma\frac{d}{dt}x(t) + [\omega_0^2 + \xi(t)]x(t) + \frac{\partial \mathcal{U}(x)}{\partial x} = 0. \qquad (1)$$

The variable $x(t)$ represents the amplitude of the oscillator at time t. The potential $\mathcal{U}(x)$ that confines the oscillator is assumed to grow faster than quadratically when $|x|\to\infty$ giving rise to nonlinear terms in the restoring force. We shall make the simplifying assumption that $\mathcal{U}$ is an even function of x and behaves as a power law of x when $|x|\to\infty$. Then, a suitable rescaling of x allows us to write

$$\mathcal{U} \sim \frac{x^{2\nu}}{2\nu} \quad \text{with} \quad \nu \geq 2. \qquad (2)$$

Typically, ν is an integer; the value $\nu=2$ corresponds to the Duffing oscillator.

The physical interpretation of Eq. (1) is that the linear stiffness of the oscillator fluctuates around its mean value ω_0^2 because of randomness in the external conditions and this randomness is represented by the external noise $\xi(t)$. We also suppose that the oscillator is subject to a linear friction with damping coefficient γ.

Equation (1) is thus a nonlinear stochastic differential equation. When the multiplicative noise $\xi(t)$ is a white noise, a coherent convention to perform stochastic calculus must be chosen. Although the Ito calculus is favored by mathematicians, we shall use here the Stratonovich calculus [27,28], which is physically more sound because it appears naturally when one considers the white noise as a limit of colored noise with very short correlation time [7,27]. This equation seems to be very elementary, but it embodies many features of random dynamics: inertial effects, nonlinear stiffness, and parametric noise. In fact, many complex dynamical systems that appear in realistic engineering problems can be reduced after some simplifying assumptions to Eq. (1). For example, the torsional stability of a suspension bridge under the influence of wind loads can be reduced to an equation similar to the one we are studying [10,29]; similarly, the dynamics of liquid sloshing, the roll motion of a ship, or the stability of helicopter rotor blades in hoover flight under atmospheric turbulence can be reduced to effective single-degree-of-freedom systems represented by a second-order equation with random parametric vibrations (for an explicit derivation of such equations, see, e.g., [10]). This equation is also akin to the model proposed by Fermi to explain the acceleration mechanism for interstellar particles; however, in the Fermi model the randomness of the acceleration results from momentum coupling (i.e., the damping γ would be random) and not from a stochastic force [30–32]. Finally, from the mathematical point of view, Eq. (1) is also very appealing. It is rich enough to exhibit an interesting dynamical behavior but simple enough to allow for explicit solutions [33–35]. This explains why such a simple model can play the role of a paradigm.

The phase-space origin $x=0$ and $dx/dt=0$ is a solution of Eq. (1). However, it can be shown that this solution is unstable [36,37] when the power spectrum of the noise contains all possible frequencies. When friction is neglected (i.e., if the underlying deterministic system is Hamiltonian) then because of the permanent injection of energy into the system by the noise, the amplitude, the velocity, and the energy undergo an anomalous diffusion. The associated anomalous diffusion exponents and amplitudes have been calculated exactly when the random excitation is a Gaussian white noise [17] or an Ornstein-Uhlenbeck process [18].

In the present work, we study the effect of the statistical properties of $\xi(t)$ on the long-time behavior of the dynamical variable $x(t)$. We must therefore specify the characteristics of the random perturbation $\xi(t)$. We shall consider a stationary Gaussian noise of zero mean value. A Gaussian process is fully characterized by its autocorrelation function defined as

$$\mathcal{S}(t'-t) = \langle \xi(t')\xi(t)\rangle. \qquad (3)$$

In Fourier space, the power spectrum of the noise is given by

$$\hat{\mathcal{S}}(\omega) = \int_{-\infty}^{+\infty} dt\, \exp(i\omega t)\mathcal{S}(t) = \int_{-\infty}^{+\infty} dt\, \exp(i\omega t)\langle \xi(t)\xi(0)\rangle. \qquad (4)$$

If $\xi(t)$ is a white noise of amplitude $\mathcal{D}$, we have

$$\mathcal{S}(t'-t)=\mathcal{D}\delta(t'-t), \quad \hat{\mathcal{S}}(\omega)=\mathcal{D}. \tag{5}$$

When $\xi(t)$ is an Ornstein-Uhlenbeck process of amplitude $\mathcal{D}$ and of autocorrelation time τ, we have

$$\mathcal{S}(t'-t)=\frac{\mathcal{D}}{2\tau}e^{-|t-t'|/\tau}, \quad \hat{\mathcal{S}}(\omega)=\frac{\mathcal{D}}{1+\omega^2\tau^2}. \tag{6}$$

In this work, we shall consider the case where the power spectrum of $\xi(t)$ decays at high frequencies in the following manner:

$$\hat{\mathcal{S}}(\omega)\sim\mathcal{D}(\omega\tau)^{-2\sigma},$$

when

$$|\omega|\to\infty. \tag{7}$$

The amplitude $\mathcal{D}$ of the noise and the correlation time τ are defined by the dimensional analogy with Eq. (6). The exponent σ characterizes the high-frequency behavior of the power spectrum. When σ is an integer, such a noise can be generated from the white noise by solving a linear differential equation of order σ.

We shall prove that in the long-time limit, the statistical properties of the oscillator in phase space can be classified by the following two parameters: (i) the exponent ν defined in Eq. (2) that encodes the large amplitude behavior of the confining potential $\mathcal{U}$ and (ii) the exponent σ that determines the high-frequency behavior of the power spectrum. For fixed values of ν and σ, the phase-space distribution takes in the long-time limit a universal form (that also depends on the dimensional parameters γ, τ, and $\mathcal{D}$) that we shall calculate.

III. EFFECTIVE DYNAMICS IN THE ASYMPTOTIC REGIME

A. Use of integrability

The mechanical energy of the oscillator is defined as

$$E=\frac{1}{2}\dot{x}^2+\mathcal{U}(x). \tag{8}$$

In the absence of noise and dissipation this quantity is conserved. This implies that the Hamiltonian system underlying Eq. (1) is integrable. It is therefore possible to define an action variable J and an angular variable ϕ so that the transformation $(p=\dot{x},x)\to(J,\phi)$ is a canonical transformation. For a given value E of the energy, the angle ϕ is given by [38,39]

$$\phi=\omega(E)\int_0^x\frac{dy}{\sqrt{2[E-\mathcal{U}(y)]}}$$

with

$$\omega(E)=\left(\frac{2}{\pi}\int_0^{x_{\max}}\frac{dy}{\sqrt{2[E-\mathcal{U}(y)]}}\right)^{-1}, \tag{9}$$

where $x_{\max}$ satisfies $\mathcal{U}(x_{\max})=E$ (recall that $\mathcal{U}$ is an even function). With this definition, the range of the phase ϕ is 2π.

The action variable J is a function of the energy only and is determined by the following equation:

$$\frac{dJ}{dE}=\frac{1}{\omega(E)}. \tag{10}$$

The Hamiltonian equations of motions, in terms of the action-angle variables, read simply as

$$\dot{J}=-\frac{dE(J,\phi)}{d\phi}=0, \tag{11}$$

$$\dot{\phi}=\frac{dE(J,\phi)}{dJ}=\omega(E). \tag{12}$$

The variables (E,ϕ) define a bona fide set of coordinates in phase space. The formulae for transforming the variables from position and velocity to energy and angle are given by

$$x(E,\phi)=\sum_{n=-\infty}^{+\infty}x_n(E)e^{in\phi}, \tag{13}$$

$$\dot{x}(E,\phi)=\sum_{n=-\infty}^{+\infty}v_n(E)e^{in\phi}=i\omega(E)\sum_{n=-\infty}^{+\infty}nx_n(E)e^{in\phi}. \tag{14}$$

Here, to write this change in variables, we have used only the deterministic and dissipationless parts of the dynamics. We choose the origin of ϕ such that $x_n(E)$ is a real number and that $x_n(E)=x_{-n}(E)$. We also write

$$\frac{1}{2}x^2(E,\phi)=\sum_{n=-\infty}^{+\infty}y_n(E)e^{in\phi}. \tag{15}$$

We then have

$$\frac{\partial}{\partial E}\left[\frac{1}{2}x^2(E,\phi)\right]=\sum_{n=-\infty}^{+\infty}\frac{dy_n(E)}{dE}e^{in\phi}, \tag{16}$$

$$\frac{d}{dt}\left[\frac{1}{2}x^2(E,\phi)\right]=i\omega(E)\sum_{n=-\infty}^{+\infty}ny_n(E)e^{in\phi}. \tag{17}$$

In the general case, we find from Eq. (1) that the time variation of the energy is given by

$$\frac{dE}{dt}=-\gamma\dot{x}^2+\xi(t)\frac{d}{dt}\left(\frac{x^2}{2}\right). \tag{18}$$

We have used here the rules of classical calculus when changing variables [7,27]. This is allowed because we are working with the Stratonovich interpretation of Eq. (1). The energy variation has thus two contributions: a loss term due to friction and a stochastic "elastic energy" term due to the work of the random multiplicative force $x\xi(t)$.

The time variation of the angle variable is given by

$$\frac{d\phi}{dt} = \omega(E) + [\gamma\dot{x} - x\xi(t)]\omega(E)\frac{\partial x}{\partial E}$$
$$= \omega(E) + \gamma\omega(E)\dot{x}\frac{\partial x}{\partial E} - \xi(t)\omega(E)\frac{\partial}{\partial E}\left[\frac{1}{2}x^2(E,\phi)\right]. \tag{19}$$

If we substitute in the two dynamical equations (18) and (19), the expressions given in Eqs. (13), (14), (16), and (17), we obtain

$$\dot{E} = \gamma\omega^2(E)\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} x_n(E)x_m(E)e^{i(n+m)\phi} - \xi(t)\omega(E)\sum_{n=-\infty}^{+\infty} iny_n(E)e^{in\phi}, \tag{20}$$

$$\dot{\phi} = \omega(E) + \gamma\omega^2(E)\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} inx_n(E)\frac{dx_m(E)}{dE}e^{i(n+m)\phi} + \xi(t)\omega(E)\sum_{n=-\infty}^{+\infty}\frac{dy_n(E)}{dE}e^{in\phi}. \tag{21}$$

We emphasize that this coupled system of stochastic nonlinear equations is rigorously equivalent to the initial random dynamical equation (1).

B. Effective Markovian description

Although the problem we study here is non-Markovian because the noise has a nonvanishing correlation time, it is possible to derive for the associated probability distribution function $P_t(E,\phi)$ an effective coarse-grained Markovian equation using a Kramers-Moyal-type expansion [7,8,27],

$$\frac{\partial P_t(E,\phi)}{\partial t} = \lim_{\delta\to 0^+}\frac{1}{\delta}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\sum_{\substack{m+k=n\\ m,k\geq 0}}\left(\frac{\partial}{\partial E}\right)^m\left(\frac{\partial}{\partial\phi}\right)^k \times[\mathbf{M}_{m,k}(E,\phi,t,\delta)P_t(E,\phi)], \tag{22}$$

where we have defined

$$\mathbf{M}_{m,k}(E,\phi,t,\delta) = \langle[\Delta E_t(\delta)]^m[\Delta\phi_t(\delta)]^k\rangle, \tag{23}$$

with

$$\Delta E_t(\delta) = E(t+\delta) - E(t) = \int_0^\delta ds\dot{E}[E(t+s),\phi(t+s),t+s], \tag{24}$$

and

$$\Delta\phi_t(\delta) = \phi(t+\delta) - \phi(t) = \int_0^\delta ds\dot{\phi}[E(t+s),\phi(t+s),t+s]. \tag{25}$$

The expressions of $\dot{E}$ and $\dot{\phi}$ on the right-hand side of Eqs. (24) and (25) are given in Eqs. (20) and (21), respectively.

The time scale δ that appears in the Kramers-Moyal expansion must be chosen in a physically relevant manner. δ has to be small but must remain larger than the intrinsic period of the oscillator (this condition is automatically fulfilled at large amplitudes because the intrinsic period tends to zero). Besides, one must also have $\delta \gg \tau$ (where τ characterizes the correlation time of the noise) in order to end up with an effective Markovian description of the dynamics.

A systematic procedure for evaluating the coefficients $\mathbf{M}_{m,k}$ that appear in Kramers-Moyal expansion has been developed by Carmeli and Nitzan in [24]. We first rewrite Eqs. (24) and (25) as

$$\Delta E_t(\delta) = \int_0^\delta ds\dot{E}[E(t)+\Delta E_t(s),\phi(t)+\Delta\phi_t(s),t+s], \tag{26}$$

$$\Delta\phi_t(\delta) = \int_0^\delta ds\dot{\phi}[E(t)+\Delta E_t(s),\phi(t)+\Delta\phi_t(s),t+s]. \tag{27}$$

The values of $\Delta E_t(\delta)$ and $\Delta\phi_t(\delta)$ are evaluated according to the following iteration scheme labeled by the integer index l:

$$\Delta E_t^{(l)}(\delta) = \int_0^\delta ds\dot{E}[E(t)+\Delta E_t^{(l-1)}(s),\phi(t)+\Delta\phi_t^{(l-1)}(s),t+s], \tag{28}$$

$$\Delta\phi_t^{(l)}(\delta) = \int_0^\delta ds\dot{\phi}[E(t)+\Delta E_t^{(l-1)}(s),\phi(t)+\Delta\phi_t^{(l-1)}(s),t+s], \tag{29}$$

with initial values given by

$$\Delta E_t^{(0)}(s) = 0, \quad \Delta\phi_t^{(0)}(s) = \omega(E)s. \tag{30}$$

Performing this expansion and neglecting the terms on the order of δ^n with $n > 1$, we obtain, in the limit $\delta \to 0$, after some systematic but tedious calculations,

$$\frac{1}{\delta}\langle\Delta E(\delta)\rangle = -\gamma\omega^2(E)\sum_{n=-\infty}^{+\infty} n^2|x_n(E)|^2 + \sum_{n=-\infty}^{+\infty}\frac{n^2\hat{\mathcal{S}}_n}{4}\left\{\frac{d[\omega(E)y_n(E)]^2}{dE} + \omega^2(E)\frac{d[y_n(E)]^2}{dE}\right\} + \frac{\omega^2(E)}{2}\sum_{n=-\infty}^{+\infty} n^2|y_n(E)|^2\frac{d\hat{\mathcal{S}}_n}{dE}, \tag{31}$$

$$\frac{1}{2\delta}\langle(\Delta E)^2(\delta)\rangle = \frac{\omega^2(E)}{2}\sum_{n=-\infty}^{+\infty} n^2|y_n(E)|^2\hat{\mathcal{S}}_n, \tag{32}$$

$$\langle\Delta E(\delta)\Delta\phi(\delta)\rangle = 0, \tag{33}$$

where we have defined

$$\mathcal{S}_n = \mathcal{S}[n\omega(E)] = \int_{-\infty}^{+\infty} dt \exp[in\omega(E)t]\langle \xi(t)\xi(0)\rangle. \quad (34)$$

We do not need to give the exact values of $\langle \Delta\phi(\delta)\rangle$ and $\langle (\Delta\phi)^2(\delta)\rangle$ because they will have no incidence in the following calculations. Higher moments are negligible at the considered order of the calculations.

C. Effective Fokker-Planck equation for the energy

We now substitute the average values calculated in Eqs. (31)–(33) into the Kramers-Moyal expansion (22). Because the cross-correlation term $\langle \Delta E(\delta)\Delta\phi(\delta)\rangle$ vanishes, we can integrate out the angular variable from Eq. (22) and obtain an effective Fokker-Planck equation for the energy,

$$\frac{\partial P_t(E)}{\partial t} = -\frac{\partial}{\partial E}\left\{\frac{\langle \Delta E(\delta)\rangle}{\delta}P_t(E)\right\} + \frac{\partial^2}{\partial E^2}\left\{\frac{\langle (\Delta E)^2(\delta)\rangle}{2\delta}P_t(E)\right\}. \quad (35)$$

Defining the following two auxiliary functions:

$$\epsilon_1(E) = \sum_{n=-\infty}^{+\infty} n^2 |x_n(E)|^2, \quad (36)$$

$$\epsilon_2(E) = \frac{1}{2}\sum_{n=-\infty}^{+\infty} n^2 |y_n(E)|^2 \hat{\mathcal{S}}_n, \quad (37)$$

we rewrite Eqs. (31) and (32) as follows:

$$\frac{1}{\delta}\langle \Delta E(\delta)\rangle = -\gamma\omega^2(E)\epsilon_1(E) + \omega^2(E)\frac{d\epsilon_2(E)}{dE} + \epsilon_2(E)\omega(E)\frac{d\omega(E)}{dE}, \quad (38)$$

$$\frac{1}{2\delta}\langle (\Delta E)^2(\delta)\rangle = \omega^2(E)\epsilon_2(E). \quad (39)$$

Substituting these expressions in Eq. (35) leads us to the effective Fokker-Planck equation for the energy,

$$\frac{\partial P_t(E)}{\partial t} = \frac{\partial}{\partial E}\left\{\omega(E)\left[\gamma\epsilon_1(E) + \epsilon_2(E)\frac{\partial}{\partial E}\right]\omega(E)P_t(E)\right\}. \quad (40)$$

For dissipationless motion $\gamma=0$, this equation does not have a stationary solution. The particle diffuses in phase space by absorbing energy from the noise and there is no mechanism to limit the growth of the amplitude. The observables grow as power laws with time as the explicit solutions of the next section will show. When $\gamma \neq 0$, the system reaches a stationary measure characterized by a nonequilibrium steady state with an asymptotic probability distribution $P_{\rm stat}(E)$ that differs from the canonical Boltzmann-Gibbs law. For $\gamma\tau \ll 1$, the effective Markovian description remains valid and the stationary solution of the effective Fokker-Planck equation is given by

$$P_{\rm stat}(E) = \frac{\mathcal{N}}{\omega(E)}\exp\left\{-\gamma\int_0^E du[\epsilon_1(u)/\epsilon_2(u)]\right\}, \quad (41)$$

where the prefactor $\mathcal{N}$ ensures the normalization of $P_{\rm stat}(E)$.

IV. EXPLICIT SOLUTIONS

A. Hamiltonian case

We shall first consider the case where the dissipation effects are not taken into account. In the absence of dissipation, the physical observables such as the amplitude, the velocity, and the energy of the oscillator grow as power laws with time. We shall calculate the associated scaling exponents and prove that their values depend only on ν, which determines the behavior of the external potential at large amplitudes and on σ that measures the relative weight of high frequencies in the noise spectrum (and which also characterizes the smoothness of the noise).

In the long-time limit, the particle diffuses to large amplitudes in phase space. Therefore, in Eq. (1), we can neglect the linear restoring force (proportional to ω_0) and replace the potential by its asymptotic behavior given in Eq. (2): $\mathcal{U}(x) \sim \frac{x^{2\nu}}{2\nu}$. Then, the change in variables to energy and angle coordinates given in Eqs. (8) and (9) takes the simpler form,

$$E = \frac{1}{2}\dot{x}^2 + \frac{x^{2\nu}}{2\nu}, \qquad \phi = \frac{\pi}{2}\frac{\int_0^{x/(2\nu E)^{1/2\nu}}\frac{du}{\sqrt{1-u^{2\nu}}}}{\int_0^1 \frac{du}{\sqrt{1-u^{2\nu}}}}. \quad (42)$$

The equation of motion for the underlying deterministic system are given by

$$\dot{E} = 0,$$

$$\dot{\phi} = \omega(E), \quad (43)$$

where

$$\omega(E) = \frac{\pi}{2\sqrt{\nu}}\frac{(2\nu E)^{(\nu-1)/(2\nu)}}{\int_0^1 \frac{du}{\sqrt{1-u^{2\nu}}}} = \mathcal{C}_\nu E^{(\nu-1)/(2\nu)}$$

with

$$\mathcal{C}_\nu = \frac{(2\nu)^{(\nu-1)/(2\nu)}\Gamma\left(\frac{\nu+1}{2\nu}\right)}{\Gamma\left(\frac{1}{2\nu}\right)}\sqrt{\pi\nu}, \quad (44)$$

where the last formula, in terms of the Euler Gamma function, is obtained from [40]. We now define [41] the hyperelliptic function $\mathcal{T}_\nu$,

$$\mathcal{T}_\nu(Y) = X \leftrightarrow Y = \int_0^X \frac{du}{\sqrt{1-u^{2\nu}}}. \quad (45)$$

The function $\mathcal{T}_\nu$ is periodic with period

$$t_\nu = 4\int_0^1 \frac{du}{\sqrt{1-u^{2\nu}}} = \frac{2\sqrt{\pi}}{\nu}\frac{\Gamma\left(\frac{1}{2\nu}\right)}{\Gamma\left(\frac{\nu+1}{2\nu}\right)}. \tag{46}$$

Inverting Eq. (42), we express the position and the velocity in terms of energy and angle using the function $\mathcal{T}_\nu$ [40,41],

$$x(E,\phi) = (2\nu E)^{1/(2\nu)}\mathcal{T}_\nu\left(\frac{t_\nu\phi}{2\pi}\right), \tag{47}$$

$$\dot{x}(E,\phi) = \sqrt{2E}\mathcal{T}'_\nu\left(\frac{t_\nu\phi}{2\pi}\right), \tag{48}$$

where $\mathcal{T}'_\nu$ is the derivative of the function $\mathcal{T}_\nu$ which, using Eq. (45), satisfies the relation

$$[\mathcal{T}_\nu(Y)]^{2\nu} + [\mathcal{T}'_\nu(Y)]^2 = 1. \tag{49}$$

The coordinates x and $\dot{x}$ are 2π periodic functions of the angle variable ϕ; they can thus be developed into Fourier series as in Eqs. (13) and (14). More precisely, if we write

$$\mathcal{T}_\nu\left(\frac{t_\nu\phi}{2\pi}\right) = \sum_{n=-\infty}^{+\infty} f_n e^{in\phi}, \tag{50}$$

we obtain the Fourier coefficients of x and $\dot{x}$

$$x_n(E) = (2\nu E)^{1/(2\nu)}f_n, \quad v_n(E) = \sqrt{2E}(inf_n). \tag{51}$$

We note that in the present case, the Fourier coefficients depend on the energy E only through a global prefactor that does not depend on the harmonic n.

This identification allows us to calculate exactly the function $\epsilon_1(E)$ defined in Eq. (36),

$$\begin{aligned}\epsilon_1(E) &= (2\nu E)^{1/\nu}\sum_{n=-\infty}^{+\infty} n^2 f_n^2 \\ &= \frac{(2\nu E)^{1/\nu}}{2\pi}\left(\frac{t_\nu}{2\pi}\right)^2\int_0^{2\pi}\left|\mathcal{T}'_\nu\left(\frac{t_\nu\phi}{2\pi}\right)\right|^2 d\phi, \\ &= \frac{(2\nu E)^{1/\nu}t_\nu}{(2\pi)^2}4\int_0^{t_\nu/4}\mathcal{T}'_\nu(Y)\sqrt{1-[\mathcal{T}_\nu(Y)]^{2\nu}}dY \\ &= \frac{(2\nu E)^{1/\nu}t_\nu}{\pi^2}\int_0^1 du\sqrt{1-u^{2\nu}},\end{aligned} \tag{52}$$

where the second equality is obtained using Parseval's identity, the third equality using Eq. (49) over a quarter of a period of the hyperellectic function $\mathcal{T}_\nu$, and the fourth equality by the change in variable $\mathcal{T}_\nu(Y)=u$. Using the expression (46) and evaluating the last integral in terms of Gamma functions [40], we obtain

$$\epsilon_1(E) = \frac{(2\nu E)^{1/\nu}\nu}{\pi\nu(\nu+1)}\frac{\Gamma^2\left(\frac{1}{2\nu}\right)}{\Gamma^2\left(\frac{\nu+1}{2\nu}\right)}. \tag{53}$$

We now calculate the function $\epsilon_2(E)$ defined in Eq. (37). Using Eqs. (15) and (47), we find that

$$y_n(E) = \frac{(2\nu E)^{1/\nu}}{2}g_n,$$

where

$$\mathcal{T}_\nu^2\left(\frac{t_\nu\phi}{2\pi}\right) = \sum_{n=-\infty}^{+\infty} g_n e^{in\phi}. \tag{54}$$

Because $\mathcal{T}_\nu$ is a real and even function of ϕ, we have $g_n = g_{-n}$ and we can rewrite $\epsilon_2(E)$ as follows:

$$\epsilon_2(E) = \frac{(2\nu E)^{2/\nu}}{4}\sum_{n=1}^{+\infty} n^2 g_n^2 \hat{\mathcal{S}}_n. \tag{55}$$

This sum depends in a nontrivial manner on the noise spectrum. We are, however, interested in the long-time behavior of the probability distribution function $\partial P_t(E)$. When $t\to\infty$, the typical value of the energy E of the system also increases without bounds and, therefore, the intrinsic frequency $\omega(E)$ of the system, which according to Eq. (44) is proportional to $E^{(\nu-1)/(2\nu)}$, also increases without bounds. Thus, when $t\to\infty$, we can replace $\hat{\mathcal{S}}_n\{=\hat{\mathcal{S}}[n\omega(E)]\}$ by its asymptotic behavior given in Eq. (7) and obtain

$$\epsilon_2(E) = \mathcal{D}\frac{(2\nu E)^{2/\nu}}{4[\omega(E)\tau]^{2\sigma}}\sum_{n=1}^{+\infty} n^{2-2\sigma}g_n^2. \tag{56}$$

Denoting by $\mathcal{A}_\sigma$ the value of the convergent series $\sum_{n=1}^{+\infty} n^{2-2\sigma}g_n^2$, we can write

$$\epsilon_2(E) = \frac{\mathcal{D}_{\nu,\sigma}}{\tau^{2\sigma}}E^{[2-\sigma(\nu-1)]/\nu} \quad \text{with} \quad \mathcal{D}_{\nu,\sigma} = \mathcal{D}\frac{(2\nu)^{2/\nu}}{4(\mathcal{C}_\nu)^{2\sigma}}\mathcal{A}_\sigma, \tag{57}$$

where $\mathcal{C}_\nu$ was defined in Eq. (44).

When the noise is white $\sigma=0$, its power spectrum is constant and the sum in Eq. (55) can be evaluated exactly. Following steps similar to those which led to Eq. (52), we obtain

$$\epsilon_2(E) = \mathcal{D}\frac{(2\nu E)^{2/\nu}}{4\pi\nu^2}\frac{\Gamma\left(\frac{1}{2\nu}\right)\Gamma\left(\frac{3}{2\nu}\right)}{\Gamma\left(\frac{\nu+1}{2\nu}\right)\Gamma\left(\frac{3\nu+3}{2\nu}\right)}. \tag{58}$$

For an Ornstein-Uhlenbeck noise $\sigma=1$ and we must evaluate the expression $\sum_{n=1}^{+\infty} g_n^2$. This, again can be done explicitly, thanks to the Parseval identity,

$$\epsilon_2(E) = \mathcal{D}\frac{(2\nu E)^{(3-\nu)/\nu}}{8\pi\nu\tau^2}\left(\frac{\Gamma\left(\frac{1}{2\nu}\right)\Gamma\left(\frac{5}{2\nu}\right)}{\Gamma\left(\frac{\nu+1}{2\nu}\right)\Gamma\left(\frac{5+\nu}{2\nu}\right)} - \frac{\Gamma^2\left(\frac{3}{2\nu}\right)}{\Gamma^2\left(\frac{\nu+3}{2\nu}\right)}\right). \tag{59}$$

We now deduce the asymptotic expression of the probability distribution function $\partial P_t(E)$ in the limit $t\to\infty$ and when there is no dissipation. The effective Fokker-Planck equation then reduces to

$$\frac{\partial P_t(E)}{\partial t} = \frac{\partial}{\partial E}\left[\omega(E)\epsilon_2(E)\frac{\partial \omega(E)P_t(E)}{\partial E}\right]. \quad (60)$$

Taking into account the expressions of $\omega(E)$ and $\epsilon_2(E)$ given in Eqs. (44) and (57), respectively, we can rewrite this equation as

$$\frac{\partial P_t(E)}{\partial t} = \frac{\mathcal{D}_{\nu,\sigma}(\mathcal{C}_\nu)^2}{\tau^{2\sigma}}\frac{\partial}{\partial E}\left(E^{\psi}\frac{\partial E^{(\nu-1)/(2\nu)}P_t(E)}{\partial E}\right)$$

with

$$\psi = \frac{\nu+3-2\sigma(\nu-1)}{2\nu}. \quad (61)$$

This equation has a self-similar structure [42] and it is natural to look for solutions of the form

$$P_t(E) = \frac{1}{E}\phi\left(\frac{E^\alpha}{Kt}\right)$$

with

$$K = \frac{\mathcal{D}_{\nu,\sigma}(\mathcal{C}_\nu)^2}{\tau^{2\sigma}}, \quad \alpha = \frac{(\sigma+1)(\nu-1)}{\nu}, \quad (62)$$

the prefactor $1/E$ ensures that $P_t(E)$ is normalized. The function $\phi(u)$ of the scaling variable $u=E^\alpha/(Kt)$ satisfies an ordinary differential equation. The solution of this equation is given by

$$\phi(u) \propto u^{(\nu+1)/(2\nu\alpha)}e^{-u/\alpha^2}. \quad (63)$$

Inserting this solution into the expression (61) for $P_t(E)$ we find, after normalization, the following asymptotic formula for the probability distribution function:

$$P_t(E) = \frac{\alpha}{\Gamma\left(\frac{\nu+1}{2\nu\alpha}\right)}\frac{1}{E}\left(\frac{E^\alpha}{\alpha^2 Kt}\right)^{(\nu+1)/(2\nu\alpha)}\exp\left(-\frac{E^\alpha}{\alpha^2 Kt}\right), \quad (64)$$

where K and α are defined in Eq. (62).

From this general result, we can retrieve the solutions for white noise (W) and for Ornstein-Uhlenbeck noise. For white noise, using Eq. (58) we have

$$P_t^{\mathrm{W}}(E) = \frac{1}{\Gamma\left(\frac{\nu+1}{2(\nu-1)}\right)}\frac{\nu-1}{\nu E}\left(\frac{E^{(\nu-1)/\nu}}{2\tilde{\mathcal{D}}_{\mathrm{W}}t}\right)^{(\nu+1)/[2(\nu-1)]} \times\exp\left\{-\frac{E^{(\nu-1)/\nu}}{2\tilde{\mathcal{D}}_{\mathrm{W}}t}\right\},$$

with

$$\tilde{\mathcal{D}}_{\mathrm{W}} = \mathcal{D}\frac{(2\nu)^{1/\nu}(\nu-1)^2}{2\nu(\nu+1)}\frac{\Gamma\left(\frac{3}{2\nu}\right)\Gamma\left(\frac{3\nu+1}{2\nu}\right)}{\Gamma\left(\frac{1}{2\nu}\right)\Gamma\left(\frac{3\nu+3}{2\nu}\right)}. \quad (65)$$

This formula was also obtained in [17] by stochastic averaging. For Ornstein-Uhlenbeck noise, we have, using Eq. (59),

$$P_t^{\mathrm{OU}}(E) = \frac{1}{\Gamma\left(\frac{\nu+1}{4(\nu-1)}\right)}\frac{2(\nu-1)}{\nu E}\left(\frac{E^{2(\nu-1)/\nu}}{2\tilde{\mathcal{D}}_{\mathrm{OU}}t}\right)^{(\nu+1)/[4(\nu-1)]} \times\exp\left\{-\frac{E^{2(\nu-1)/\nu}}{2\tilde{\mathcal{D}}_{\mathrm{OU}}t}\right\},$$

with

$$\tilde{\mathcal{D}}_{\mathrm{OU}} = \frac{\mathcal{D}}{4\tau^2}\left(\frac{\nu-1}{\nu}\right)^2(2\nu)^{2/\nu} \times\left(\frac{\Gamma\left(\frac{\nu+1}{2\nu}\right)\Gamma\left(\frac{5}{2\nu}\right)}{\Gamma\left(\frac{1}{2\nu}\right)\Gamma\left(\frac{5+\nu}{2\nu}\right)} - \frac{\Gamma^2\left(\frac{\nu+1}{2\nu}\right)\Gamma^2\left(\frac{3}{2\nu}\right)}{\Gamma^2\left(\frac{1}{2\nu}\right)\Gamma^2\left(\frac{\nu+3}{2\nu}\right)}\right). \quad (66)$$

This formula is identical to the one derived in [18] by using the averaging method at the second order.

To summarize, we have shown that in the long-time limit, the scaling behavior of the dissipationless nonlinear oscillator in the presence of a noise with a power spectrum that satisfies Eq. (7) is given by

$$E \sim \left(\frac{\mathcal{D}t}{\tau^{2\sigma}}\right)^{\nu/[(\sigma+1)(\nu-1)]},$$

$$x \sim \left(\frac{\mathcal{D}t}{\tau^{2\sigma}}\right)^{1/2[(\sigma+1)(\nu-1)]},$$

$$\dot{x} \sim \left(\frac{\mathcal{D}t}{\tau^{2\sigma}}\right)^{\nu/2[(\sigma+1)(\nu-1)]}. \quad (67)$$

These scalings derived here by a systematic calculation agree with the results given in [20] that were conjectured by performing a partial resummation of the small correlation-time expansion of the stochastic Liouville equation associated with the random dynamical system under study. The method used in [18,20] was an approximation that could not be applied to the system studied here but only to a simplified model. In fact, the resummation technique yielded the correct exponents but the prefactors were out of reach. Here, we have obtained the closed expression (64) for the probability distribution function, which contains the full information on the statistics of the system when $t\to\infty$, i.e., it provides us the scaling exponents and the corresponding prefactors.

B. Dissipative case

In the presence of dissipation, the system reaches in the long-time limit a steady state with the stationary probability given by Eq. (41). Using Eqs. (53) and (56), we find the explicit formula for this stationary distribution,

$$P_{\mathrm{stat}}(E) = \frac{\alpha}{\Gamma\left(\frac{\nu+1}{2\nu\alpha}\right)} \frac{1}{E}\left(\frac{2\nu\gamma E^{\alpha}}{\alpha K(\nu+1)}\right)^{(\nu+1)/(2\nu\alpha)} \times \exp\left(-\frac{2\nu\gamma E^{\alpha}}{\alpha K(\nu+1)}\right), \tag{68}$$

where K and α are defined in Eq. (62). From this stationary PDF, we find that the behavior of the mean energy as a function of the parameters of the model is given by

$$E^{\alpha} \sim \frac{\mathcal{D}}{\gamma\tau^{2\sigma}}. \tag{69}$$

We observe that the expression of $P_{\mathrm{stat}}(E)$ becomes identical to that given in Eq. (64) if the time t in Eq. (64) is replaced by t_γ with

$$t_\gamma = \frac{1}{\gamma}\frac{\nu+1}{2(\sigma+1)(\nu-1)}. \tag{70}$$

The value of t_γ determines the time scale at which the system becomes sensitive to the dissipative effects. For $t \ll t_\gamma$, the system evolves as if it were Hamiltonian and the physical observables grow algebraically with time. For $t \gg t_\gamma$, the system sets in its steady state and the statistical averages of the physical observables become stationary. Finally, we justify the validity range of the stationary probability distribution (68). We recall [24] that the Markovian approximation is valid only if γ^{-1} is much smaller than the correlation time τ of the noise, i.e.,

$$\gamma\tau \ll 1. \tag{71}$$

Besides, the presence of dissipation should not alter significatively the dynamics of the fast variable ϕ; thus, on the right-hand side of Eq. (21) the second term must remain much smaller than the first one. Using Eqs. (44), (47), and (48) this requires that $\gamma \ll E^{(\nu-1)/2\nu}$. From the typical value (69) of the energy and the expression of α given in Eq. (62), this condition becomes

$$(\gamma\tau)^{2\sigma}\gamma^3 \ll \mathcal{D}. \tag{72}$$

Thus, a sufficient condition is $\gamma^3 < \mathcal{D}$. For higher values of the dissipation rate, the expression (68) will no more be valid. The behavior of the system can change drastically and a phase transition to a state localized at the origin $x=\dot{x}=0$ can occur [3,36,43].

C. Additive noise case

In this last subsection, we study the nonlinear oscillator driven by an additive noise,

$$\frac{d^2}{dt^2}x(t) + \gamma\frac{d}{dt}x(t) + \frac{\partial\mathcal{U}(x)}{\partial x} = \xi(t). \tag{73}$$

The change in coordinates to energy and angle variables is the same as in Eqs. (13) and (14). The dynamical equations in these coordinates will read as

$$\dot{E} = \gamma\omega^2(E)\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} x_n(E)x_m(E)e^{i(n+m)\phi} + \xi(t)\omega(E)\sum_{n=-\infty}^{+\infty} inx_n(E)e^{in\phi}, \tag{74}$$

$$\dot{\phi} = \omega(E) + \gamma\omega^2(E)\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} inx_n(E)\frac{dx_m(E)}{dE}e^{i(n+m)\phi} - \xi(t)\omega(E)\sum_{n=-\infty}^{+\infty}\frac{dx_n(E)}{dE}e^{in\phi}. \tag{75}$$

Using the Carmeli-Nitzan technique, we derive the following effective Fokker-Planck equation for the energy:

$$\frac{\partial P_t(E)}{\partial t} = \frac{\partial}{\partial E}\left\{\omega(E)\left[\gamma\epsilon_1(E) + \tilde{\epsilon}_2(E)\frac{\partial}{\partial E}\right]\omega(E)P_t(E)\right\}, \tag{76}$$

the function $\epsilon_1(E)$ was defined in Eq. (36) and $\tilde{\epsilon}_2(E)$ is given by

$$\tilde{\epsilon}_2(E) = \frac{1}{2}\sum_{n=-\infty}^{+\infty} n^2|x_n(E)|^2\hat{\mathcal{S}}_n. \tag{77}$$

Denoting by $\tilde{\mathcal{A}}_\sigma$ the value of the convergent series $\sum_{n=1}^{+\infty} n^{2-2\sigma}f_n^2$, we can write

$$\tilde{\epsilon}_2(E) = \frac{\tilde{\mathcal{D}}_{\nu,\sigma}}{\tau^{2\sigma}}E^{[1-\sigma(\nu-1)]/\nu} \quad \text{with } \tilde{\mathcal{D}}_{\nu,\sigma} = \mathcal{D}\frac{(2\nu)^{2/\nu}}{4(\mathcal{C}_\nu)^{2\sigma}}\tilde{\mathcal{A}}_\sigma. \tag{78}$$

The constant $\mathcal{C}_\nu$ was defined in Eq. (44). It is possible to carry out explicit calculations following the same lines as for the multiplicative noise case. If dissipation is neglected, the probability distribution function is found to be

$$P_t(E) = \frac{\tilde{\alpha}}{\Gamma\left(\frac{\nu+1}{2\nu\tilde{\alpha}}\right)}\frac{1}{E}\left(\frac{E^{\tilde{\alpha}}}{\tilde{\alpha}^2\tilde{K}t}\right)^{(\nu+1)/(2\nu\tilde{\alpha})}\exp\left(-\frac{E^{\tilde{\alpha}}}{\tilde{\alpha}^2\tilde{K}t}\right), \tag{79}$$

where $\tilde{\alpha}$ and $\tilde{K}$ are given by

$$\tilde{\alpha} = \frac{(\sigma+1)(\nu-1)+1}{\nu}, \quad \tilde{K} = \frac{\tilde{\mathcal{D}}_{\nu,\sigma}(\mathcal{C}_\nu)^2}{\tau^{2\sigma}}. \tag{80}$$

For the special cases of white noise or Ornstein-Uhlenbeck noise, explicit expressions for $\tilde{K}$ can be derived and the formulae for the probability distribution function derived in [19] using the averaging method are recovered.

Thus, in the absence of dissipation, the following algebraic scalings for the main observables of the system are satisfied:

$$E \sim \left(\frac{\mathcal{D}t}{\tau^{2\sigma}}\right)^{\nu/[(\sigma+1)(\nu-1)+1]},$$

$$x \sim \left(\frac{\mathcal{D}t}{\tau^{2\sigma}}\right)^{1/[2(\sigma+1)(\nu-1)+2]},$$

$$\dot{x} \sim \left(\frac{\mathcal{D}t}{\tau^{2\sigma}}\right)^{\nu/[2(\sigma+1)(\nu-1)+2]}. \quad (81)$$

In particular, when the noise is white (or if the potential is quadratic), we recover the result that the energy grows linearly with time. The amplitude of the noise being constant, the exponents in the additive case are smaller than those in the multiplicative case, as expected.

If we take dissipation into account, the system reaches a steady state in the long-time limit. The expression of the stationary probability matches that of Eq. (79) if t is taken to be

$$t \rightarrow \frac{1}{\gamma} \frac{\nu+1}{2[\nu+\sigma(\nu-1)]}. \quad (82)$$

This expression defines the dissipation time scale for a nonlinear oscillator subject to an additive noise.

V. CONCLUSION

We have used an effective coarse-grained Markovian description to carry out a quantitative analysis of the nonlinear oscillator confined by a polynomial potential and subject to Gaussian noise of arbitrary spectrum that decays as a power law at high frequencies. This approach has allowed us to calculate the distribution, in phase space, of the dynamical system in the long-time limit. In the absence of dissipation, the particle diffuses without bounds with an anomalous scaling law. The energy transfer from the random perturbation to the particle also follows a scaling power-law. The diffusion exponents and the corresponding amplitudes are determined exactly. In the presence of dissipation, the system reaches a nonequilibrium steady state; the corresponding stationary distribution has also been calculated analytically in the limit of vanishingly small dissipation. The advantage of the method used here as compared to stochastic averaging is that it can readily be adapted to any noise spectrum. For white and Ornstein-Uhlenbeck noises, the two approaches give identical results. It would be of great interest to apply this method to higher dimensional integrable systems subject to stochastic perturbations, such as the nonlinear Schrödinger equation in a random potential, which is often used to study the effect of nonlinearity on localization.

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